Hamiltonian for Long-range Beam-Beam interactions – Fourier Expansion Coefficients

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Abstract

In [1] the Lie-algebraic method was used to develop generalized Courant-Snyder invariant in the presence of an arbitrary number of beam-beam collisions, head-on or long-range, in a storage ring collider. Each beam-beam collision point is described by a set of Fourier coefficients, computed numerically. This note presents analytic expressions for these coefficients.

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I. INTRODUCTION

The long-range beam-beam Hamiltonian divided by $\frac{N_b r_0}{\gamma}$ can be written as:

$$H = \int_0^P (1-e^{-\alpha})\frac{d\alpha}{\alpha} = \tag{1}$$

$$= \gamma + \Gamma_0(P) + ln(P), \tag{2}$$

where

$$P = P(x) \equiv \frac{1}{2}\left((n_x + \frac{x}{\sigma})^2 + n_y^2\right). \tag{3}$$

Here x is the particle coordinate; d_x the real space offset of the collision point in x direction and similar for y direction. Also $n_{x,y} = d_{x,y}/\sigma$ are the normalized offsets. Also $\sigma = \sqrt{\epsilon\beta}$ with β being the beta function and ϵ the emittance.

In [1], we used the form (2) to calculate numerically the coefficients c_m (m=integer) in the Fourier expansion of H. Our goal here is to derive analytical expressions for these coefficients. For this we use the form (1), i.e. we follow the approach in [2], where the same is done for the case of head-on collisions ($n_x = n_y = 0$). Not surprisingly, the result is that c_m can be written as series of Bessel-I functions. In the case of $n_x = n_y = 0$ these reduce to the single Bessel-I function of [2],

II. SOME PROPERTIES OF H

H can be written in terms of several mutually related special functions – Appendix A.

In the form (2), $\Gamma_s(P) \equiv \Gamma(s, P)$ denotes the upper incomplete gamma function – see Abramowitz-Stegun.

The form (1) is identical to the usual definition of $Ein(P)$. Further, $Ein(z)$ is related to the exponential integral $E_1(z)$, so one may also write:

$$H = Ein(P) = \gamma + lnP + E_1(P).$$

From (2), we have the relations:

$$\frac{\partial H}{\partial P} = \frac{1}{P} - \frac{e^{-P}}{P} \tag{4}$$

and

$$\frac{\partial P}{\partial x} = \frac{1}{\sigma^2}(x + n_x\sigma), \tag{5}$$

so that the long-range beam-beam kick is:

$$
\begin{aligned}
f(x) &\equiv \frac{d}{dx}H(x) = \frac{\partial H}{\partial P}\frac{dP}{dx} = \\
&= \frac{2(x+n_x\sigma)}{(x+n_x\sigma)^2+(n_y\sigma)^2}\left(1-e^{-\frac{(x+n_x\sigma)^2+(n_y\sigma)^2}{2\sigma^2}}\right).
\end{aligned}
$$

The nonlinear part of the Hamiltonian is $H - H^{(1)} - H^{(2)}$, where $H^{(1)} \sim x$ and $H^2 \sim x^2$. The linear in x part of H is

$$
\begin{aligned}
H^{(1)} &= \left(\frac{\partial H(x,\sigma)}{\partial x}\right)|_{x=0}\ x = \\
&= \frac{2n_x}{\sigma(n_x^2+n_y^2)}\left(1-e^{-\frac{n_x^2+n_y^2}{2}}\right)\ x. \qquad (6)
\end{aligned}
$$

The non-linear in x part of H, i.e. closed orbit subtracted is

$$
H^{non} = H - H^{(1)}. \qquad (7)
$$

Further, an alternative way to write the integral is (by taking $\alpha = tP$ in (1)):

$$
\begin{aligned}
H &= \int_0^P (1-e^{-\alpha})\frac{d\alpha}{\alpha} = \\
&= \int_0^1 (1-e^{-tP})\frac{dt}{t}. \qquad (8)
\end{aligned}
$$

III. ACTION-ANGLE VARIABLES

For simplicity, let's assume that there is no offset in the orthogonal plane, i.e. $n_y = 0$. All expressions we get can easily be modified to account for $n_y \neq 0$. In addition, we know that its effect is small.

Introduce action angle variables as in [2]

$$
\begin{aligned}
x &= \sqrt{2A\beta}\ \sin\phi = \\
&= \sqrt{2A/\epsilon}\,\sigma\ \sin\phi = \\
&= n_\sigma\ \sigma\ \sin\phi, \qquad (9)
\end{aligned}
$$

Here

$$
n_\sigma = \sqrt{2A/\epsilon} = \sqrt{2A\beta/\sigma^2}. \qquad (10)
$$

With our simplifying assumption $n_y = 0$, from (3) we have:

$$
P = \frac{1}{2}\,(n_x + \frac{x}{\sigma})^2 = \frac{1}{2}\,(n_x + n_\sigma \sin\phi)^2 \qquad (11)
$$

and corresponding H defined by (2) or (8). Also from (6)

$$H^{(1)} = \frac{2}{n_x}\left(1 - e^{-\frac{n_x^2}{2}}\right) n_\sigma \sin\phi. \tag{12}$$

We have defined P, H and $H^{(1)}$ as functions of amplitude A and phase ϕ. The dependence on A is implicit – through n_σ: $P(n_\sigma, \phi)$, $H(n_\sigma, \phi)$ and $H^{(1)}(n_\sigma, \phi)$. The particle amplitude is measured in number of sigmas n_σ wrt the the closed orbit, which orbit is to first order just the the IP offset at this location, n_x sigmas from the axis. Most interesting is the case (dynamic aperture) when both these quantities are large: $n_x \sim 9$ and $n_\sigma \sim 7 - 10$.

IV. FOURIER EXPANSION

The expansion of the Hamiltonian is in the form:

$$H = c_0 + \sum_{m \neq 0} c_m e^{im\phi} \tag{13}$$

and we need to compute the Fourier coefficients

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi} H(n, \phi) d\phi, \quad \text{m=integer}. \tag{14}$$

Assuming this is done, we can replace H in (13) with $H^{non} - H^{(1)}$ which will lead to modification of only two coefficients: $c_{\pm 1}$ which can easily be deduced from (12).

A. Direct numerical way

Numerical calculation of c_m can be done as in [1]. By taking the second form (2) of H and also P from (11) and $H^{(1)}$ from (12), we have:

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\left(\gamma + \Gamma_0(P) + ln(P) - H^{(1)}\right) d\phi. \tag{15}$$

```
cm[m_, nx_, n_] :=
  Module[{P, Hnon},
   P := (nx^2/2 + n nx Sin[\[Phi]] + 1/2 n^2 Sin[\[Phi]]^2);
   Hnon := EulerGamma + Gamma[0, P] + Log[P] -
   cosub  2/nx (1 - E^(-(nx^2/2))) n Sin[\[Phi]];
   1/(2 \[Pi]) NIntegrate[Exp[-I m \[Phi]] Hnon , {\[Phi], 0, 2 \[Pi]},
```

```
      AccuracyGoal -> 8] // Chop];
cosub=1;
m = 1;
nx = 9;
n = 5;
cm[m, nx, n]/I^m


-0.0511155
```

In this example $m = 1, n_\sigma = 5, n_x = 9$, the closed orbit is subtracted ($\texttt{cosub} = \texttt{1}$) and the result is $c_1 = -0.0511155\, i$.

B. Analytical way

Let us take the first form of H (1) , with integration variable t as in (8), and substitute it into (14). Consider first the expression under the integral over t:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\,(1 - e^{-tP(n_\sigma,\phi))})d\phi, \tag{16}$$

We rewrite the exponent factor as

$$\begin{aligned} -tP(n_\sigma,\phi) &= -t\left(\frac{1}{2}n_x^2 + n_\sigma n_x \sin\phi + \frac{1}{2}n_\sigma^2 \sin^2\phi\right) \\ &= -\frac{t}{2}n_x^2 - tn_\sigma n_x \sin\phi - \frac{t}{2}n_\sigma^2 \sin^2\phi = \\ &= z_3 - z_1 \sin\phi + 2z_2 \sin^2\phi, \end{aligned} \tag{17}$$

with the temporary notations $z_1 = tn_\sigma n_x$, $z_2 = -\frac{t}{4}n_\sigma^2$ and $z_3 = -\frac{t}{2}n_x^2$. The exponents that appear in (16) can be expressed through the modified Bessel functions I_k:

$$e^{-z_1\ \sin\phi} = \sum_{k=-\infty}^{\infty} i^k e^{ik\phi} I_k(z_1) = \tag{18}$$

$$= I_0(z_1) + 2\sum_{k=0}^{\infty} \cos k(\phi + \frac{\pi}{2}) I_k(z_1) \tag{19}$$

and

$$e^{\,2z_2\,\sin^2\phi} \;=\; e^{z_2}\sum_{k=-\infty}^{\infty}(-1)^k e^{2ik\phi}I_k(z_2) = \tag{20}$$

$$= \; e^{z_2}\left(I_0(z_2)+2\sum_{k=0}^{\infty}(-1)^k\cos 2k\phi\; I_k(z_2)\right) \tag{21}$$

The m-th Fourier component of the product of (19) and (21) is

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}e^{-z_1\,\sin\phi}e^{2\,z_2\,\sin^2\phi} =$$
$$= e^{z_2}\sum_{q,k=-\infty}^{\infty}(-1)^k i^q I_q(z_1)\;I_k(z_2)\;\delta(2k+q-m) =$$
$$= e^{z_2}\sum_{k=-\infty}^{\infty} i^m I_{m-2k}(z_1)I_k(z_2). \tag{22}$$

By using (17) and (22), the expression (16) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\left(1-e^{-tP(n_\sigma,\phi)}\right)d\phi =$$
$$= 1-e^{z_3}e^{z_2}\sum_{k=-\infty}^{\infty} i^m I_{m-2k}(z_1)I_k(z_2) =$$
$$= \begin{cases} 1-e^{z_3}e^{z_2}\sum_{k=-\infty}^{\infty} I_{-2k}(z_1)I_k(z_2), & \text{if } m=0 \\ -e^{z_3}e^{z_2}\sum_{k=-\infty}^{\infty} i^m I_{m-2k}(z_1)I_k(z_2), & \text{if } m\neq 0. \end{cases} \tag{23}$$

The expansion (23) still needs to be divided by t and integrated over t from 0 to 1. Upon replacing z_i with their values it is easy to see that for (23) there will be no singularities at $t=0$ (use the asymptotes of I for small argument). For the $m=0$ term we get:

$$c_0 \;=\; \int_0^1\frac{dt}{t}\left(1-e^{-\frac{t}{2}n_x^2}e^{-\frac{t}{4}n_\sigma^2}\sum_{k=-\infty}^{\infty} I_{-2k}(tn_\sigma n_x)I_k(-\frac{t}{4}n_\sigma^2)\right). \tag{24}$$

This is the averaged H over the unperturbed phase and it can be used to compute the tune shift $-1/(2\pi)dc_0/dA$. The formula for n_σ is (9). Also, $\frac{dn_\sigma}{dA}=\frac{1}{\epsilon n_\sigma}$.

For c_m, with $m\neq 0$:

$$c_m \;=\; \int_0^1\frac{dt}{t}\left(-e^{-\frac{t}{2}n_x^2}e^{-\frac{t}{4}n_\sigma^2}\sum_{k=-\infty}^{\infty} i^m I_{m-2k}(tn_\sigma n_x)I_k(-\frac{t}{4}n_\sigma^2)\right) =$$
$$= \; -i^m\sum_{k=-\infty}^{\infty}\int_0^1\frac{dt}{t}e^{-\frac{t}{2}n_x^2}e^{-\frac{t}{4}n_\sigma^2}I_{m-2k}(tn_\sigma n_x)I_k(-\frac{t}{4}n_\sigma^2). \tag{25}$$

shows a Mathematica implementation of the function $c_m(n_\sigma,n_x)$ in (25), represented by `CM[m_, nx_, n_]`. This example uses the same values as for Exhibit1 above and the result is the same.

```
CM[m_, nx_, n_] :=
(cosub KroneckerDelta[Abs[m] - 1] (I)^(m) 1/nx (1 - E^(-(nx^2/2))) n -
   I^m Sum[NIntegrate[(Exp[-t nx^2/2] Exp[-t/4 n^2] BesselI[m - 2 k,
         t n nx] BesselI[k, -t/4 n^2])/t, {t, 0, 1}], {k, -kmax, kmax}])//Chop
kmax=20;
cosub=1;
m = 1;
nx = 9;
n = 5;
CM[m, nx, n]/I^m

-0.0511155
```

For a collision point without offset, by taking $n_x = 0$ in (25) and using the property of I for zero argument $I_{m-2k}(0) = \delta_{m-2k}$ we get:

$$c_m = \begin{cases} \int_0^1 \frac{dt}{t}\left(1 - e^{-\frac{t}{4}n_\sigma^2} I_0(-\frac{t}{4}n_\sigma^2)\right), & \text{if m=0} \\ -\int_0^1 \frac{dt}{t} e^{-\frac{t}{4}n_\sigma^2} I_{m/2}(\frac{t}{4}n_\sigma^2), & \text{if m=even} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

From (9) we have $n_\sigma^2/4 = \frac{A\beta}{2\sigma^2}$, By using also $I_0(-z) = I_0(z)$ and (8), we see that this is the same as the expression given by Chao [2].

[1] D.Kaltchev, W.Herr Analytical calculation of the smear for long-range beam-beam interactions, PAC09 get paper

[2] A. Chao, Truncated Power Series Algebra, lectures

V. APPENDIX

Appendix A: H via special-functions

The Hamiltonian expressed through the function $\gamma^*(s,z)$**, see Abramowitz-Stegun**:

$$H(x) = -\frac{\partial\gamma^*}{\partial s}|_{s\to 0} = \int_{P(x)}^{\infty} \frac{e^{-t}}{t} dt = \gamma + \Gamma_0(P) + ln(P)$$
$$P(x) = \frac{1}{2}\left((n_x + \frac{x}{\sigma})^2 + n_y^2\right).$$

The Hamiltonian expressed through the exponential integral Ei(x). For real, nonzero values of x, the exponential integral Ei(x) can be defined as

$$E_i(x) = \int_{-\infty}^{x} e^t \frac{dt}{t}. \tag{A1}$$

The integral has to be understood in terms of the Cauchy principal value, due to the singularity in the integrand at zero. In general, a branch cut is taken on the negative real axis and Ei can be defined by analytic continuation elsewhere on the complex plane. For complex values of the argument, this definition becomes ambiguous due to branch points at 0 and ∞. In general, a branch cut is taken on the negative real axis and Ei can be defined by analytic continuation elsewhere on the complex plane. One uses $E_1(z)$ defined as:

$$E_1(z) = \int_z^{\infty} e^{-t}\frac{dt}{t} = \int_1^{\infty} (e^{-tz})\frac{dt}{t} = \int_0^1 e^{-z/u}\frac{du}{u}, \quad \text{Re(z)} \geq 0 \tag{A2}$$

Both Ei and E_1 can be written more simply using the entire function Ein,such that

$$E_1(z) = -\gamma - lnz + E_{in}(z). \tag{A3}$$

The Hamiltonian is $H(x) = Ein(P)$, where:

$$Ein(z) = \int_0^z (1 - e^{-t})\frac{dt}{t}, \tag{A4}$$

or also

$$\begin{aligned} Ein(z) &= \gamma + lnz + E_1(z) = \\ &= \gamma + lnz + \int_z^{\infty} e^{-t}\frac{dt}{t} = \\ &= \gamma + lnz + \int_1^{\infty} (e^{-tz})\frac{dt}{t} = \\ &= \gamma + lnz + \int_0^1 e^{-z/u}\frac{du}{u}, \quad \text{Re(z)} \geq 0. \end{aligned} \tag{A5}$$