Analytical calculation of the smear for long-range beam-beam interactions

Dobrin Kaltchev, TRIUMF, Vancouver, Canada
Werner Herr, CERN, Geneva, Switzerland

Abstract

The Lie-algebraic method is used to develop generalized Courant-Snyder invariant in the presence of an arbitrary number of beam-beam collisions, head-on or long-range, in a storage ring collider. The invariant is obtained by concatenating nonlinear beam-beam maps in the horizontal plane and to first order in the beam-beam parameter. Tracking evidence is presented to illustrate that with LHC parameters the invariant is indeed preserved and can be used to predict the smear of horizontal emittance observed in tracking simulations. We discuss the limits of applicability of this model for realistic LHC collision schemes.

INTRODUCTION

The beam-beam interaction, in the weak-strong model, can be studied with transfer maps. Assuming the weak-beam particle motion to consist of passages through the strong beam (collisions) alternating with linear-motion sections, the ring transfer map is given by the product:

$$M = \prod_{k=1}^{N_{\text{IP}}} e^{:F^{(k)}:} e^{:F_2^{(k)}:} = e^{:h:},$$

where N_{IP} is the number of collision points, head-on or long-range, $e^{:F^{(k)}:}$ and $e^{:F_2^{(k)}:}$ are Lie operators associated with a linear matrix and a beam-beam kick respectively and h is a generalized Courant-Snyder invariant. By using the properties of Lie operators for F-factors in action-angle representation it is not difficult to derive an expression for h to first order of the beam-beam parameter in the horizontal plane. For multiple *head-on* interaction points in the LHC such an expression was reported previously [1],[2], where also a condition was derived for suppression of resonances. In this paper we present an invariant h valid for both head-on and long-range collisions and verify with tracking that in absence of other lattice perturbations h is indeed preserved. We compare the distortion of h with the smear observed in Sixtrack [5] simulations and discuss the limits of agreement of such a first-order theory with tracking.

For a ring with a single ($N_{\text{IP}} = 1$) head-on collision point, both the nonresonant and resonant h have been analyzed by Dragt [3] (see also Chao's lectures [4]).

INVARIANT FOR MULTIPLE COLLISION POINTS

Fourier expanded potential

In the LHC, a particle moving in the horizontal plane sees both horizontal (near IP5) and vertical (near IP1) head-on and long-range collisions. Assuming round Gaussian beam profile at all collision points the kick is: $\Delta x' = \lambda f$, where

$$f(x) = \frac{2(x + d_x)}{(x + d_x)^2 + d_y^2} \exp\left[-\frac{(x + d_x)^2 + d_y^2}{2\sigma^2}\right], \quad (1)$$

$\lambda \equiv \frac{N_b r_0}{\gamma}$, N_b is the number of particles per bunch, r_0 – the classical particle radius, γ - the relativistic parameter and β, $\sigma = \sqrt{\epsilon\beta}$, $d_{x,y}$ are the beta function, the transverse beam size and the transverse separations at the location of the kick; ϵ is the emittance.

The beam-beam potential is $\lambda F(x) = \lambda \int_0^x f(x')d\,x'$. We first transform $F(x)$ to action-angle coordinates (A, ϕ) by substituting in it $x = \sqrt{2A}\sigma \sin\phi$ and then expand it in Fourier series. The result is:

$$F(A, \phi) = \int_0^1 \frac{dt}{t}\left(1 - e^{-t\left[\left(\sqrt{A}\sin\phi + \frac{\tilde{d}_x}{\sqrt{2}}\right)^2 - \frac{\tilde{d}_y^2}{2}\right]}\right) =$$

$$= \sum_{n=-\infty}^{\infty} c_n(A)\, e^{in\phi}, \quad (2)$$

where $\tilde{d}_{x,y} \equiv d_{x,y}/\sigma$. The complex coefficients satisfy $c_n^* = c_{-n}$ and will be found numerically [1]. If $\tilde{d}_{x,y} = 0$ (head-on collision) then c_n are expressed via Bessel functions [4]. In what follows we will also need the linear ($F_1 \sim x$) and quadratic ($F_2 \sim x^2$) parts of the potential:

$$F_1 = \frac{2\sqrt{2A}\sin\phi}{\tilde{d}^2}\tilde{d}_x\left(1 - \exp\frac{-\tilde{d}^2}{2}\right), \quad (3)$$

$$F_2 = \frac{2A\sin^2\phi}{\tilde{d}^4} \times$$

$$\times \left[-\tilde{d}_x^2 + \tilde{d}_y^2 + (\tilde{d}_x^2 + \tilde{d}_x^4 - \tilde{d}_y^2 + \tilde{d}_x^2\tilde{d}_y^2)\exp\frac{-\tilde{d}^2}{2}\right],$$

where $\tilde{d}^2 \equiv \tilde{d}_x^2 + \tilde{d}_y^2$.

Nonresonant invariant h

Using the Campbell-Baker-Hausdorff formula it can be shown [1] that the quantity h is given by:

$$\frac{1}{\mu}\,h(A, \phi) = -A + \sum_{k=1}^{N_{\text{IP}}} \frac{\lambda^{(k)}}{\epsilon}\tilde{h}^{(k)}; \quad (4)$$

$$\tilde{h}^{(k)} = c_0^{(k)} + \sum_{n=1}^{m}\frac{(-1)^n n}{2\sin\frac{n\mu}{2}}\left[c_n^{(k)}\, e^{in(\frac{1}{2}\mu - \mu^{(k)} - \phi)} + c.c.\right];$$

$$c_n^{(k)}(A) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}\, F^{(k)}(A, \phi)\, d\phi \quad (n = 0, \dots, m).$$

---

[1] The integral in (2) can be expressed through the incomplete gamma function: $F(A, \phi) = \gamma + \Gamma[0, P] + \log P$, where $P = \frac{1}{2}(\tilde{d}_x^2 + \tilde{d}_y^2) + \sqrt{2A}\,\tilde{d}_x\,\sin\phi + A\sin^2\phi$ ($\gamma = 0.577216$ is the Euler's constant).

Here $\tilde{h}^{(k)}$ is the contribution to the invariant of the k-th collision point located at betatron phase $\mu^{(k)}$, while $\mu = 2\pi Q_x$ (Q_x is the ring tune). The Lie-factor $F^{(k)}$ is given by (2) with $\tilde{d}_{x,y}$ replaced with $\tilde{d}_{x,y}^{(k)}$ – normalized separations at the k-th collision point. The number of Fourier harmonics taken (m) is adjusted to correspond to nine significant figure accuracy. The factors $\lambda^{(k)}$ allow to model lumped collisions as in [6], [7] (not used here).

Since the coefficients $c_n^{(k)}(A)$ only weekly depend on A, they can be computed only once for, say, the initial (first turn) value of A. This claim is verified in the next section. We see then that the contribution of each collision point to the distortion of phase space at some amplitude is defined by an unique set of coefficients. This set depends on the normalized separations at this collision point.

VERIFICATION OF THE INVARIANT

This section presents numerical evidence that h defined with (4) is indeed preserved for some sample (minimal) set of long-range collisions in the LHC: 12 long-range interactions at locations with separations 9.4, 8.5, and 5 sigma positioned symmetrically on both sides of IP5 and IP1.

Comparison with a simple tracking model

The model tracking is based on kicks $f^{(k)}(x) - f^{(k)}(0)$, where $f^{(k)}(x)$ is as in (1), alternating with unperturbed linear matrices. The procedure is to iterate an initial condition $(A_0, \phi_0) = (\frac{n_\sigma^2}{2}, \pi/2)$ thus getting a sequence of points A_i, ϕ_i, where i is the turn number, and then compare the quantities A_i and $h(A_i, \phi_i)$. Figure 1, analogous to one in [3], shows that h is indeed more constant than the ordinary Courant-Snyder invariant $(-A)$. The coefficients $c_n^{(k)}(A_0)$ in (4) are computed from potentials with subtracted linear term: $F^{(k)} - F_1^{(k)}$.
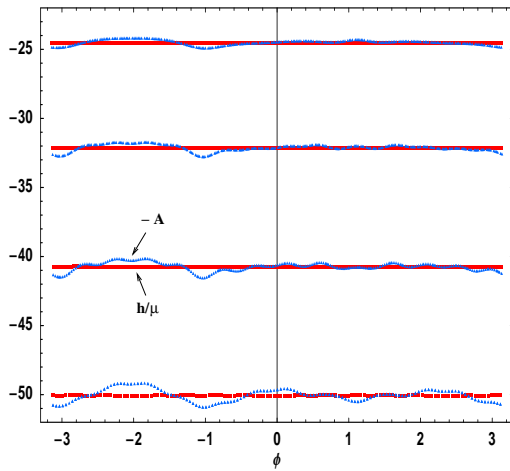

Figure 1: Plots of the quantities h/μ (red) and $(-A)$ (blue) for increasing starting amplitude $x = n_\sigma\sigma$; $n_\sigma = 7, 8, 9, 10$. Large variations of $(-A)$ occur for larger initial betatron amplitude.

To compute smear from the invariant we use the relation $h(A, \phi) = h(A_0, \phi_0)$ and express A as a function of ϕ. Figure 2 illustrates that A_i, ϕ_i lay on the resultant theoretical curve $A(\phi)$. By covering the curve $A(\phi)$ with some dense equidistant set of points, smear is defined as the r.m.s. deviation of the A-values from this set divided by the mean.
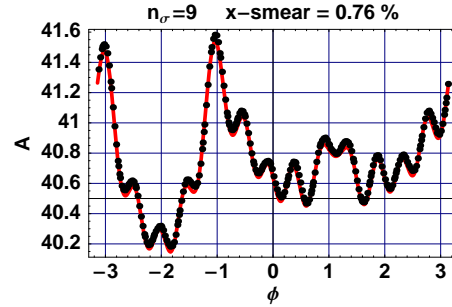

Figure 2: Same as case $n_\sigma = 9$ on Fig. 1, but a different way to show the agreement (as in [1]). Red: theoretical $A(\phi)$ dependence used to define smear. Black: model tracking.

Comparison with Sixtrack

For comparison with Sixtrack the action is defined via linearly-perturbed β function and correspondingly $c_n^{(k)}$ are computed from potentials with subtracted linear and quadratic parts: $F^{(k)} - F_1^{(k)} - F_2^{(k)}$. See Figure 3. The agreement is shown on Figure 4.
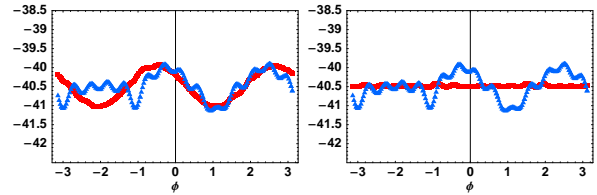

Figure 3: Same as case $n_\sigma = 9$ on Fig. 1, but invariant computed from linearly perturbed β at the initial point for tracking. Left: potential used is $F - F_1$. Right: the potential is $F - F_1 - F_2$ (in this case h is more constant).
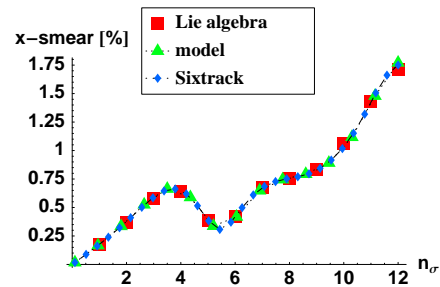

Figure 4: Agreement between Lie algebra, tracking model and Sixtrack for a sample set of long-range collisions in the LHC. Shown is smear of the horizontal emittance computed over 2^{10} turns as a function of initial $x = n_\sigma\sigma$.

REALISTIC LHC COLLISION SCHEME

Limits of applicability

Figure 5 allows to estimate the limits of our first order Lie-algebra model. Two scenarios are considered: the case of 25 ns bunch separation, corresponding to one head-on and 30 long-range collisions near each of the two main interaction points IP5 and IP1, and the 50-ns case when the number of long-range collisions is about twice smaller. Inspection of Figure 5 shows that the model is able to predict the smear to amplitude 12σ for intensities up to 1/2 of nominal for the 50 ns scenario, and to about 1/4 of the nominal for the 25 ns case. If the bunch population is larger than these limits, then for amplitudes above $6\,\sigma$ maps of higher order are needed.
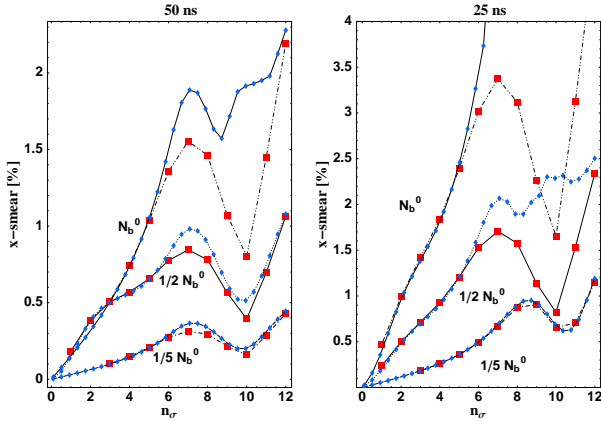

Figure 5: Agreement between Lie-algebra model (red) and Sixtrack (blue) for increasing bunch population N_b and two LHC collision schemes "50 ns" and "25 ns" (head-on is included). The curves correspond to $1, 1/2$ and $1/5$ times the nominal bunch population $N_b^0 = 1.15\ 10^{11}$.

Individual contributions of long-range collision points

It is of interest to compare the individual terms $\tilde{h}^{(k)}(A, \phi)$ in Eqn. 4. Each such term describes the contribution of a single collision point to distortion of the ordinary Courant-Snyder invariant and hence to smear. One example is shown on Figure 6 for the "25-ns" case of Figure 5 (right), initial amplitude $6\,\sigma$ and $N_b = 1/2 N_b^0$. Only long-range collisions near IP5 (30 in total) are included since both the contribution from IP1 and the total contribution from head-on events are small. The top plot corresponds to nominal values of the separations while the bottom one, to the early separation scheme [7]. In the latter case, horizontal smear is dominated by contributions (light blue) of collisions in immediate vicinity of IP5, which change their $\tilde{d}_x$ value from ± 9.4 to $\pm 5\sigma$.
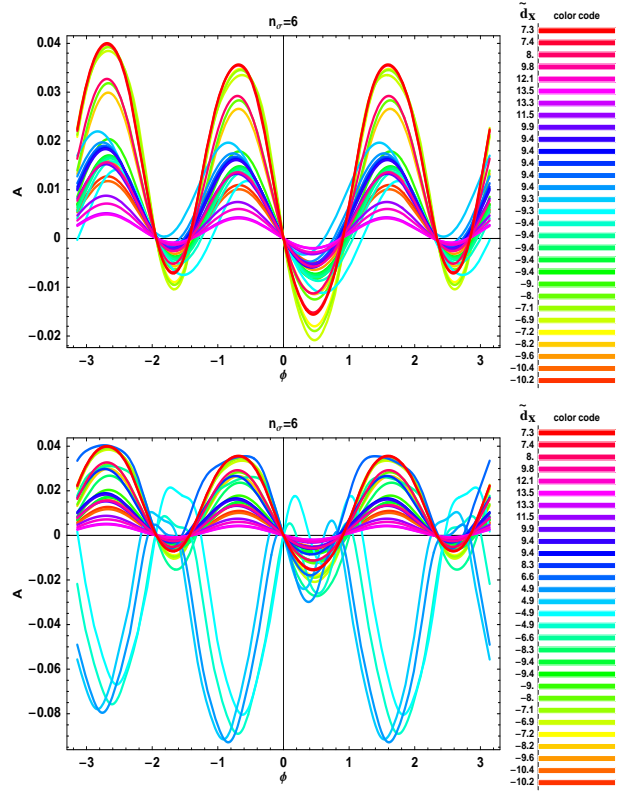

Figure 6: Individual contributions to h of the 30 long-range collisions near IP5 (horizontal crossing) for two scenarios: nominal (top) and early separation [7] (bottom).

REFERENCES

[1] W. Herr, D. Kaltchev, Effect of phase advance between interaction points in the LHC on the beam-beam interactions, LHC Project Report 1082, Apr 1 2008

[2] D. Kaltchev , On beam-beam resonances observed in LHC tracking, TRI-DN-07-9.

[3] Alex J. Dragt, Oleg G. Jakubowicz, Analysis of the Beam-Beam Interaction Using Transfer maps, Proceedings of the Beam Beam Interaction Seminar, Stanford Ca, May 22-23, 1980, SLAC-R-541

[4] A. Chao, Lie Algebra Techniques for Nonlinear Dynamics, http:www.slac.stanford.edu/~achaok th

[5] F. Schmidt, SixTrack, V.4.2 Single Particle Tracking Code Treating Transverse Motion with Synchrotron Oscillations in a Symplectic Manner, CERN/SL/94-56, Jul 2008.

[6] Y. Papaphilippou, F. Zimmermann, Weak-strong beam-beam simulations for the LHC, PRST Vol. 2, 104001 (1999).

[7] Sterbini, G Koutchouk, Jean-Pierre, An Early Beam Separation Scheme for the LHC Luminosity Upgrade EPAC 2006, Edinburgh and LHC-PROJECT-Report-972.