FOURIER COEFFICIENTS OF LONG-RANGE BEAM-BEAM HAMILTONIAN VIA TWO-DIMENSIONAL BESSEL FUNCTIONS*

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Abstract

The two-dimensional coefficients (resonance basis) in the Fourier expansion of the long-range beam-beam Hamiltonian have been expressed through the (little known) generalized modified Bessel functions of two arguments. We present a procedure to compute these coefficients in the above representation. The method is applied to the nominal scenario HL-LHC lattice and benchmarked against MadX simulations of detuning.

INTRODUCTION

The Fourier-expansion coefficients of the accelerator Hamiltonian appear naturally in analytical calculations of amplitude detuning, low-order normal form and related to it driving terms. In studies of long-range beam-beam interaction with neglected bunch-length effects, i.e. a two-dimensional Hamiltonian $H(x, y)$, the coefficients c_{mk} are usually expressed through modified Bessel functions of the first kind $I_n(u)$, or their relatives [4], and given as single integrals over sum of the products of two $I_n(u)$ [1–3]. High ~ 40 orders n are needed – [1], [2].

In this paper c_{mk} are expressed through generalized Bessel functions depending on two real variables [5], [6]:

$$\mathbf{I}_n(u, v) \equiv \sum_{q=-\infty}^{\infty} I_{n-2q}(u) I_q(v), \tag{1}$$

These $\mathbf{I}_n(u, v)$ have properties very similar to the above ordinary Bessel $I_n(u)$, but are much less familiar. Equivalently, one may use the functions $\mathbf{\Lambda}_n(u_1, u_2) = e^{-u_1-u_2}\mathbf{I}_n(u_1, u_2)$, which possess similar properties. Either kind forms resonance basis for the above Hamiltonian deemed to be more-natural than the one based on their ordinary single-argument counterparts.

The formulae used below for generating function, recursion and derivatives of $\mathbf{I}$ and $\mathbf{\Lambda}$ have been derived by transforming the results in [6] – a paper devoted to two-dimensional analogues of $J_n(x)$. For $\mathbf{\Lambda}$ no references have been found.

We will derive relations between c_{mk} and $\mathbf{I}$ (or $\mathbf{\Lambda}$) focusing on applications to HL-LHC, i.e. case of large separations and long-range collision points with unequal sigmas of strong and weak beam ($r \neq 1$) – this may also be relevant to wire compensation of beam-beam [1]. Numerical procedure in `fortran` has been developed that relies on precomputed and stored $\mathbf{\Lambda}$ functions. In the last Section, analytic amplitude detuning is benchmarked against MadX tacking in the HL-LHC lattice.

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[1] S. Fartoukh, private communication

HAMILTONIAN COEFFICIENTS VIA TWO-ARGUMENT BESSEL FUNCTIONS

When written in terms of unperturbed action-angle variables, the Hamiltonian H(x,y) describing the beam-beam kick at a head-on (HO), or long-range (LR) interaction point (IP) depends on a_z, d_z (z=x,y)– normalized test-particle amplitude and full separation at this IP. We assume round-beam optics and equal emittances of weak and strong beam, but possibly "flat-beam" long-range IP, i.e. one with $\beta_x \neq \beta_y$, so that $r \equiv \frac{\sigma_y}{\sigma_x} \neq 1$. In this latter case, let us use the symmetry of Interaction Region 5 (IR5), where the beams are separated in x direction. Here weak and strong-beam sigmas are related by $\sigma^w_x = \sigma_y$ $\sigma^w_y = \sigma_x$, so that: $x = r\sigma_x a_x \sin\phi_x$, $y = \frac{\sigma_y}{r} a_y \sin\phi_y$. Thus, for $r \neq 1$, the formulae below are valid for IR5, while IR1 (vertical separation) can be treated symmetrically. The case $r = 1$ is generic (the formulae are valid for any insertion).

For an IP in IR5 the Hamiltonian, in units of $\frac{N_b r_0}{\gamma}$, is:

$$\begin{aligned}
H(x, y) &= \int_0^1 \frac{dt}{tg(t)}[1 - e^{-t(P_x+P_y)}];\\
P_z &\equiv \frac{1}{2}\left(\bar{d}_z + \bar{a}_z \sin\phi_z\right)^2,\\
\bar{a_x} &= ra_x,\ \ \bar{d_x} = d_x,\ \ \bar{a_y} = \frac{a_y}{g(t)},\ \ \bar{d_y} = \frac{rd_y}{g(t)},
\end{aligned}$$

where γ is the relativistic factor, N_b is the bunch population and $g(t) \equiv \sqrt{1 + (r^2 - 1)\,t}$. Removing the bar in all variables gives the generic case of round-beam IP ($r = 1$, $g = 1$). By expanding P_z, we have the relations:

$$\begin{aligned}
-tP_z &= -u_1^{(z)} \sin\phi_z + 2u_2^{(z)} \sin^2\phi_z + u_3^{(z)} \qquad (2)\\
u_1^{(z)} &= t\bar{a_z}\bar{d_z},\ \ u_2^{(z)} = -\frac{t}{4}\bar{a_z}^2,\ \ u_3^{(z)} = -\frac{t}{2}\bar{d}_z^2,\\
u_{23}^{(z)} &\equiv u_2^{(z)} + u_3^{(z)} = -\frac{t}{2}(\bar{a_z} - \bar{d_z})^2 - u_1^{(z)} - u_2^{(z)}.
\end{aligned}$$

Using Eqn. (2), the Fourier coefficients

$$c_{mk} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} He^{-im\phi_x - ik\phi_y} d\phi_x d\phi_y$$

are expressed [1], [2], [8] as integrals over Bessel series (the Introduction). Somewhat more directly, let us combine (2) with the generating function for two-argument Bessel functions:

$$e^{-u_1\ \sin\phi_z + u_2(2\sin{\phi_z}^2 - 1)} = \sum_{k=-\infty}^{\infty} i^k \mathbf{I}_k(u_1, u_2) e^{ik\phi_z}. \tag{3}$$

The result is (here $\delta = 1$ if $m = k = 1$ and 0 otherwise):

$$c_{mk} = \int_0^1 \frac{dt}{tg(t)}\left[\delta - \ \mathbf{Q}_m^{(x)}(t)\ \mathbf{Q}_k^{(y)}(t)\right] \tag{4}$$

The $\boldsymbol{Q}$s can be written either in terms of $\mathbf{I}$, or $\mathbf{\Lambda}$:

$$\mathbf{Q}_m^{(z)}(t) \equiv i^m e^{u_{23}^{(z)}} \mathbf{I}_m(u_1^{(z)}, u_2^{(z)}) = \quad (5)$$
$$= i^m e^{-\frac{t}{2}(\bar{a_z}-\bar{d}_z)^2} \mathbf{\Lambda}_m(u_1^{(z)}, u_2^{(z)}). \quad (6)$$

Barring small differences in notation, and the fact that for $r \neq 1$ the IR5 symmetry has already been embedded, this is identical to [2], see e.g. Eqn 52. Notice that the first form (5) contains no barred variables while the second (6) does, but is more intuitive: for in-plane LR collision the exponent factor is just the squared distance $a_x - |d_x|$ between the weak-beam particle amplitude and the strong-beam centroid.

Particular cases follow directly. E.g. for in-plane LR collision one uses that in the y-plane $\mathbf{I}_0(0,0) = 1$, to get: $c_m \equiv c_{m,0} = \int_0^1 \frac{dt}{t}(\delta(m) - K(t))$, where $K(t) = i^m e^{-\frac{t}{4}a_x^2} e^{-\frac{t}{2}d_x^2} \mathbf{I}_{m,0}(ta_x d_x, -\frac{t}{4}a_x^2)$.

RECURSIVE PROPERTIES

The functions $\mathbf{I}_m(u_1, u_2)$ obey one recursion

$$u_1[\mathbf{I}_{m-1} - \mathbf{I}_{m+1}] + 2u_2[\mathbf{I}_{m-2} - \mathbf{I}_{m+2}] = 2m\mathbf{I}_m, \quad (7)$$

and two derivative properties:

$$\frac{\partial \mathbf{I}_m}{\partial u_1} = \frac{1}{2}[\mathbf{I}_{m-1} + \mathbf{I}_{m+1}]\,; \; \frac{\partial \mathbf{I}_m}{\partial u_2} = \frac{1}{2}[\mathbf{I}_{m-2} + \mathbf{I}_{m+2}] \quad (8)$$

(and a similar one for $\mathbf{\Lambda}$). On the other hand, by rewriting (4) using (5) the coefficients c_{mk} are:

$$\int_0^1 \frac{dt}{tg(t)}\left[\delta - i^{m+k} e^{u_{23}^{(x)}+u_{23}^{(y)}} \mathbf{I}_m(u_1^{(x)}, u_2^{(x)}) \mathbf{I}_k(u_1^{(y)}, u_2^{(y)})\right] \quad (9)$$

Hence the higher-order resonance coefficients are not independent. For fixed arguments u_1 and u_2, (7) allows to find easily $\mathbf{I}_m$ for all m, given the first four ($\mathbf{I}_m$ for $m = 0, 1, 2, 3$) – useful also in numerical calculations (see below). A recursive procedure for c_{mk} has been found – albeit rather difficult to solve, so not used in numerical calculations – where the complication arising from $\mathbf{I}_m$ being under an integral sign is compensated by the additional differential relations (8). Our (preliminary) conclusion is that at least in principle, the c_{mk} can all be expressed through the ones of order up to and including order $m = 3$ ("octupole"), assuming a beam-beam "multipole" has been defined in the new resonance basis in a way similar to "usual" multipoles.

The following conjecture is then made. If, as a result of lumped correction, *local*, i.e. at this IP, compensation of all terms to order $m = 3$ has taken place, then all resonance terms are also canceled. On the other hand, as we will see, (8) allows to express amplitude dependent tune-shifts using the first three ($\mathbf{I}_m$ for $m = 0, 1, 2$). To summarize, if terms up to order 2 ("sextupole") are locally minimized, then the footprint is reduced. If in addition the $m = 3$ term is minimized, then this leads to all resonance terms being small. Same or similar conclusions have been made in [7].

FOOTPRINT

The nonlinear detunings with amplitude ΔQ_x, ΔQ_y are given by the partial derivatives of c_{00} over the actions J_x, J_y. By setting $m = k = 0$ in (4) and replacing δ with unity:

$$c_{00} = \int_0^1 \frac{dt}{t\,g(t)}\left[1 - \mathbf{Q}_0^{(x)}(t)\mathbf{Q}_0^{(y)}(t)\right];$$
$$\frac{\partial c_{00}}{\partial a_x} = -\int_0^1 \frac{dt}{t\,g(t)} \frac{\partial \mathbf{Q}_0^{(x)}(t)}{\partial a_x} \mathbf{Q}_0^{(y)}(t) \quad (10)$$
$$\text{(and similar for y);}$$
$$\Delta Q_z = 2\xi \frac{1}{a_z}\frac{\partial c_{00}}{\partial a_z}, \text{ where z = x or y.} \quad (11)$$

Here $\xi \equiv \frac{N_b r_0}{4\pi\gamma\epsilon}$ is the beam-beam parameter (both H and $c_{0,0}$ are in units of $\frac{N_b r_0}{\gamma}$) and we have used $\frac{da_z}{dJ_z} = -\frac{1}{\epsilon a_z}$. According to (11) one needs to compute (10) twice, where under the integral there is the product of Q and a derivative of Q (with $x \leftrightarrow y$). The derivative can be taken using a property as (8). The result, in terms of $\mathbf{\Lambda}$, is:

$$\mathbf{Q}_0^{(z)} = e^{-\frac{t}{2}(\bar{a_z}-\bar{d}_z)^2}\mathbf{\Lambda}_0,$$
$$\frac{\partial \mathbf{Q}_0^{(z)}}{\partial a_z} = \eta_z e^{-\frac{t}{2}(\bar{a_z}-\bar{d}_z)^2}\left[-\frac{\bar{a_z}}{2}[\mathbf{\Lambda}_0 + \mathbf{\Lambda}_2] + \bar{d}_z\mathbf{\Lambda}_1\right]$$
$$\eta_x \equiv rt, \quad \eta_y \equiv t/g(t);$$
$$\mathbf{\Lambda}_{0,1,2} \equiv \mathbf{\Lambda}_{0,1,2}(u_1^{(z)}, u_2^{(z)}). \quad (12)$$

As advertised, the footprint depends on the first three $\mathbf{\Lambda}$. Finally notice that in (10), since $1/t$ cancels and $g > 0$ for any r, there is no singularity under the integral.

Familiar expression for single-plane head-on, IP without offset, follow from $I_{m-2q}(0) = \delta(m - 2q)$, or alternatively from $\mathbf{I}_m(0, u_2) = I_{m/2}(u_2)$ (only even m remain).

The tune-shift expressions derived in [3] ($r = 1$ only) follow from (10) by replacing in it $\mathbf{\Lambda}$ with its generating function form (3), reversing the order of integration and using the fact that for $r = 1$ (only!) the integral over t is solvable.

NUMERICAL IMPLEMENTATION AND COMPARISON WITH MADX

For numerical calculations of both c_{mk} and detuning we have encoded the two-dimensional Bessel functions either in *Mathematica* (any Bessel arguments), or as a `fortran` code (faster). In `fortran`, we take advantage of the recursion property: since all c_{mk} depend on only four two-argument functions $\mathbf{I}_n$, these are precomputed and stored as four tables. Thus only cases n=0,1,2,3 need be computed to high accuracy as the rest of $\mathbf{I}_n$ follow recursively. The integral over t is taken using bi-linear approximation of these tables.

In what follows, our goal is to verify the Hamiltonian by comparing expressions (11), (12) with MadX tracking (`dynaptune`) and prove overall applicability of the method at nominal HL-LHC settings, i.e. large Bessel arguments, summing range $q_{\max}$ and size of the prestored `fortran` arrays.

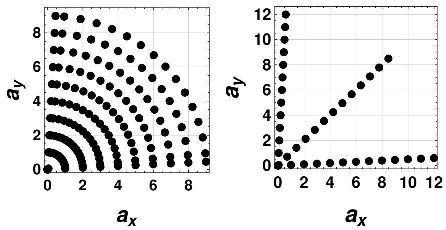
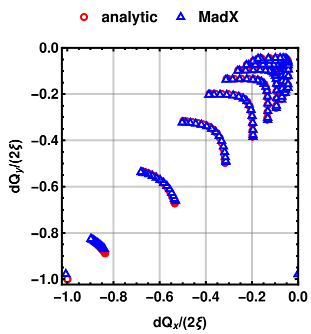


Figure 1: a_z ranges – 15 angles (left) and 3 angles (right).

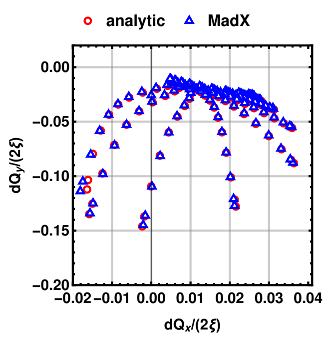
1) a_z as Fig 1 left

d_x	d_y	σ_x [mm]	r
0.	0.	0.007	1.
0.	0.	0.007	1.



2) a_z as Fig 1 left

d_x	d_y	σ_x [mm]	r
6.236	0.	0.176	1.



3) a_z as Fig 1 left

d_x	d_y	σ_x [mm]	r
12.472	0.	0.176	1.

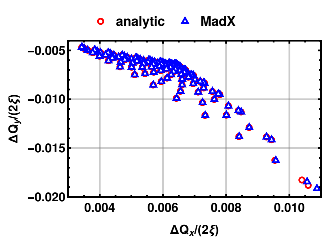


Figure 2: Sample IPs with $r = 1$: **1)** head-on IP1 and 5; **2)** single long range IP closest to IP5, `bbip5L1` for half the nominal crossing angle (required $q_{\max}$=25); **3)** same as **2)**, but full nominal crossing angle (required $q_{\max}$=35)

a_z as Fig 1 right

d_x	d_y	σ_x [mm]	r
-9.376	0.	2.391	0.499

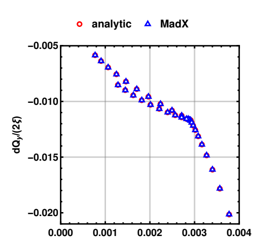


Figure 3: IP with r=0.5: nominal HL-LHC (`bbip5R10`)

All examples are made with *Mathematica*, however for amplitudes $a_z < 7$ the `fortran` code is able to reproduce all plots, except for Fig. 4. We use HL-LHC with round-beam optics at IP1(5), $\beta^{\star}$= 15 cm, full crossing angle 295 μrad angle, normalized emittance $\epsilon_{\rm norm} = 2.5$ μm, $N_b = 1 \times 10^{11}$ (ξ= 0.00488). Tracking is for $\sim 10^3$ turns, on-momentum, with beam-beam being the only nonlinearity. The agreement between MadX tracking and the detuning formula (12) is demonstrated on Fig. 2 ($r = 1$), Fig. 3 ($r < 1$) and Fig. 4 ($r > 1$) with the beam-beam setup and initial amplitude range referring to Fig. 1 as indicated on the left.

a_z as Fig 1 right

d_x	d_y	σ_x [mm]	r
18.768	0.	1.193	2.003

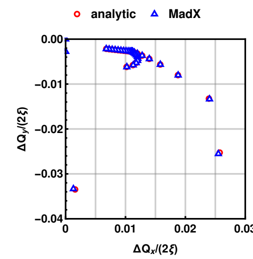


Figure 4: IP with r=2: nominal HL-LHC (`bbip5L10`)

The nominal HL-LHC setup implies very small tune-shifts per long-range IP, hence the need of high accuracy – the relative difference ($\Delta Q/Q$) between tracking and analytic formula is predominantly better than 4×10^{-4}. For each plot, before the comparison is made, tiny tune shifts $\sim 5 \times 10^{-5}$ still present in the beam-beam free lattice are subtracted from the MadX output. For case **3** on Figure 2 the $q_{\max}$ had to be increased from 25 to 35 – otherwise the last two red circles at the bottom would deviate substantially. Largest Bessel arguments correspond to the setup in Fig. 4: maximum a_z =12, d_s $\sim$ 19 with r=2. In this last case the *Mathematica*'s computing time was $\sim$ 300 sec.

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