Analysis of long range studies in the LHC - comparison with the model

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Abstract

We find that the observed dependencies (scaling) of long-range beam-beam effects on the beam separation and intensity are consistent with the simple assumption that, all other parameters being the same, the quantity preserved during different setups is the first order smear as a function of amplitude.

INTRODUCTION

Proposed method

In several Machine Development (MD) studies, see [1] and the references therein, reduced crossing angles have been used to enhance long-range beam-beam effects and thus facilitate their measurement. The basic assumption taken in this paper is that under such conditions a single nonlinearity, the one caused by beam-beam, dominates the dynamics. Hence the method followed: we choose some simple low-order dynamical quantity that characterizes phase space distortion and assume that when this quantity is the same, the behavior of the system is the same. A most obvious candidate is the first-order smear – r.m.s. deviation of the phase-space ellipse from the perfect one. At a fixed amplitude smear is defined as the averaged generalized Courant-Snyder invariant over the angle variable [8].

An analytical expression was previously found [8] for the smear $S$ as a function of amplitude $n_\sigma$. Suppose that the parametric dependence of $S(n)$ on several beam-beam related parameters: relativistic $\gamma$, number of particles per bunch $N_b$, crossing angle $\alpha$ and normalized separations $n_{l.r.}$ is known. According to the assumption above for two machine configurations $a$ and $b$ one should have:

$$S(n_\sigma; N_{b_a}^{a}, n_{l.r.}^{a}, \alpha^{a}, \gamma^{a}) = S(n_\sigma; N_{b_b}^{b}, n_{l.r.}^{b}, \alpha^{b}, \gamma^{b}). \quad (1)$$

As a particular application of (1), we considered two experiments where the intensities are $N_{b_a}^{a}$ and $N_{b_b}^{b}$. All other parameter being the same, given $\sigma^a$, one can compute the expected $\sigma^b$. Our task will be to show that the result agrees with observations.

Analytical calculation of invariant and smear

Our derivation of $S(n_\sigma)$ is based on the Lie-algebraic method – concatenation of Lie-factor maps, and is valid only to first order in the beam-beam parameter and in one-dimension, either in horizontal or in vertical plane, but for an arbitrary distribution of beam-beam collisions, head-on or long-range, around the ring.

For a ring with single head-on collision point, Hamiltonian perturbation analysis of the beam-beam interaction without or with crossing angle has been done by a number of authors, mostly in the resonant case. Nonlinear invariants of motion, both nonresonant and resonant, were analyzed by Dragt [2] with the one-turn map as observed immediately after the kick being nonresonant, were analyzed by Dragt [2] with the one-turn map as observed immediately after the kick being resonant, were analyzed by Dragt [2] with the one-turn map as observed immediately after the kick.

Detailed derivation of $h$ to first order in the beam-beam perturbation strength can be found in A. Chao’s lectures:

$$h(J, \phi) = -\mu J + \sum_{n=-\infty}^{\infty} c_n^{(ho)}(J) \frac{n \mu}{2 \sin h_{l.r.}} e^{in(\phi + \mu/2)}, \quad (4)$$

where $\mu$ is the ring phase advance and $c_n^{(ho)}(J)$ are coefficients in the Fourier expansion of $F$, when the latter is rewritten in action-angle coordinates $J, \phi$. The coefficients are shown to be related to the modified Bessel functions. Analytic expressions for the invariant $h$, first-order smear and second order detuning for the case of nonlinear multipole kicks distributed in an arbitrary way around the ring were derived by Irvin and Bengtsson [4]. Smear, the distortion of the ideal phase-space ellipse, is formally defined in [6]. Finally, notice that extracting the smear is a natural step in the procedure that brings the map into its normal form, [5].

In [7], following the Lie algebraic procedure in [3] and [4] we generalized (4) to describe multiple head-on kicks (IP1 and IP5) for the case of LHC. In [8] an expression was presented valid for arbitrary number of head-on (h.o.) and long-range (l.r.) collisions. This expression, to be derived in detail next, has been used on several occasions to interpret results from Sixtrack simulations.

DERIVATION OF THE INVARIANT

Multiple collision points

The horizontal betatronic motion of a weak-beam test particle depends on its initial amplitude $n_\sigma$ (in units of $\sigma$)
and the collision set: a set of all h.o. and l.r. collisions, a.k.a. interaction points (IPs), that this particle sees over a single revolution. Let us label the set with an index $k$, limiting ourselves to only IPs located within the main interaction regions IR5 (horizontal crossing) and IR1 (vertical crossing). In case of $50$ ns bunch spacing, $k$ ranges from $1$ to $34$ which includes $32$ long-range IPs ($N_{l.r.} = 32$).

The Lie map depends on the above defined collision set through the normalized separations $n_x(k) = d_{x_k} / \sigma(k)$ and the unperturbed horizontal betatron phases $\phi(k)$ at the IPs. Here $d_{x_k}$ is the real-space offset of the strong beam centroid in $x$ or $y$ direction and it has been assumed that both the weak- and strong-beam transverse distributions are round Gaussians of the same rms, i.e.:

$$
\sigma(k) = \sqrt{\beta(k)} \epsilon \quad (\beta(k) = \beta_x(k) \equiv \beta(k)).
$$

In (5), $\beta(k)$ are the beta functions and $\epsilon$ is the emittance. It will be shown below that off-plane collisions contribute very little to smear, thus after excluding these the problem becomes one-dimensional and may easily be illustrated, Figure 1. Here $n_x(k)$ are the strong-beam centroids in amplitude space: points $(s(k), n_x(k))$ with $s$ being the distance to IP5 in meters.

![Figure 1: Schematic view of weak and strong beam trajectories in real (top) and amplitude (bottom) spaces. A reduced set is used: $N_{l.r.} = 8+8 = 16$ (k=1,18). Beam-beam Hamiltonian]

For a single collision, (2), by omitting the superscript $k$ in $\sigma$ and $n_{x,y}$, the $x$-motion is described by a kick-factor $F$ (or Hamiltonian $H$) [8]

$$
F = -H(x) = \int_0^P (1 - e^{-\alpha}) \frac{d\alpha}{\alpha} = \pi + \Gamma_0(P) + \ln(P),
$$

$$
P = P(x) = \frac{1}{2} \left[ (n_x + \frac{\bar{x}}{\sigma})^2 + n_y^2 \right],
$$

where $F$ is in units of $\lambda \equiv N_{x,y} / 2\sigma$, $r_0$ is the classical particle radius, $\Gamma_0(P) \equiv \Gamma(s, P)$ denotes the upper incomplete gamma function [9] and $\bar{\gamma} = 0.577216$ is the Euler’s constant. The corresponding beam-beam kick is:

$$
\Delta x' = \frac{dF}{dt}(x) = \frac{\partial F}{\partial P} \frac{dP}{dt} = \left[ 2e - \frac{(n_x + \sigma n_y)^2 + (n_y \sigma)^2}{\sigma^2} \right].
$$

The Fourier expansion of $H$ is:

$$
H(n_x, \phi) = \sum m C_m e^{im\phi},
$$

where $C_m \equiv \frac{2\pi}{2\pi} \int_0^{2\pi} e^{-im\phi} H d\phi$. These coefficients are easily computed numerically by using the implementation of $\Gamma$ in Mathematica, [8]. Further, analytic expressions in the form of single integrals over Bessel functions have been derived in [11]. We display these again in the simplified case $n_y = 0$ (no off-plane collisions):

$$
C_m|_{n_y=0} = \int_0^1 \frac{dt}{t} \times
\left[ 1 - e^{-\frac{1}{4}n_x^2} - e^{-\frac{1}{4}n_y^2} \sum k = -\infty \infty I_{2k}(t n_x n_y) I_k(-\frac{t}{4} n_x^2) \right]
$$

if $m = 0$ and

$$
- e^{-\frac{1}{4}n_x^2} - e^{-\frac{1}{4}n_y^2} \sum k = -\infty \infty t^m I_{m-2k}(t n_x n_y) I_k(-\frac{t}{4} n_x^2)
$$

if $m \neq 0$.

In the head-on case ($n_x(k) = n_y(k) = 0$) the coefficients $c_m$ reduce to the $c_m^{(1)}$ from [3]. Notice that in the most interesting case: amplitudes near the dynamic aperture, both $n_x$ and $n_y$ and hence the Bessel function arguments are large ($\gg 1$).

Our first step is to remove the linear and quadratic parts $F_{(2)} = \frac{\partial F}{\partial t} \big|_{t=0} x$ and $F_{(2)} = \frac{\partial^2 F}{\partial t^2} \big|_{t=0} x^2$. The nonlinear kick-factor and corresponding kick are:

$$
F_{\text{nonl}} = F - F_{(1)} - F_{(2)},
$$

$$
\Delta x_{\text{nonl}} \equiv \frac{dF}{dt}(F_{\text{nonl}}(x)).
$$

As a next step, we rewrite (10) in action-angle coordinates $J, \phi$ by substituting in it $x = \sqrt{2J} \sin \phi = n_x \sigma \sin \phi$, where $n_x = \sqrt{2T} = \sqrt{2J/\epsilon}$ is the test particle amplitude (see Eq. A.1 in Appendix). Next, we expand in Fourier series:

$$
F_{\text{nonl}}(n_x \sigma \sin \phi) = c_0 + \sum_{m=0}^\infty c_m e^{im\phi}.
$$

The coefficients $c_m$ are naturally the same as $C_m$ above, with the exception of $c_1$ and $c_2$ which contain additional $\sin$ and $\sin^2$ terms, see Eq. A.1.

### Lie map and invariant

For an arbitrary set of collisions $n_x(k), \phi(k)$ ($k = 1, N$) we represent the LHC lattice by a combination of linear
elements and nonlinear kicks. It is shown in the Appendix that, to first order in $\lambda$, the Lie map has the same form as the one for a single kick (2), where however the factor $F$ is given by the sum:

$$ F \equiv \sum_{k=1}^{N} F^{(k)}_{\text{non}}(n_{\sigma}, \phi) $$

and $F^{(k)}_{\text{non}}$ are such that, compared to (11), the $k$th IP participates with a phase shifted by $\phi^{(k)}$.

$$ F^{(k)}_{\text{non}}(n_{\sigma}, \phi) \equiv F^{(k)}_{\text{non}}(x) \bigg|_{x = n_{\sigma} \sigma^{(k)} \sin(\phi^{(k)})} = \sum_{m \neq 0} C^{(k)}_{m} e^{im\phi}. $$

(12)

The shift in phase means that the coefficients in (12) are simply related to $c_{m}^{(k)}$, $C_{m}^{(k)} \equiv c_{m}^{(k)} e^{im\phi^{(k)}}$ and still satisfy $C_{-m} = C_{m}^{*}$. Another important property of the expansion is that only the oscillating part is taken (the $m=0$ term is excluded). The invariant for multiple collision points is (Appendix):

$$ h(I, \phi) = -\mu J - \lambda \sum_{k=1}^{N} \sum_{m=1}^{\infty} \frac{m \mu \epsilon m(I)}{2 \sin(\frac{m\pi}{2})} e^{im(\phi + \mu/2 + \phi^{(k)})} + c.c. $$

The surface of section in phase space is given by $h(I, \phi) = \text{const}$. A natural initial condition is now imposed: that the initial point in phase space for a particle starting at $x_0 = n_{\sigma} \sigma$, i.e. with an amplitude $I_0 = I_{0}/\epsilon = n_{\sigma}^{2}/2$, lies on the curve representing the invariant:

$$ h(I, \phi) = h(I_{0}, \pi/2). $$

(13)

For a fixed $I_{0}$, this equation defines implicitly $I$ as a function of $\phi$. It satisfies the initial condition $I(0) = I_{0}$:

$$ I(\phi) = I_{0} + \sum_{k=1}^{N} \left( dI^{(k)}(\phi) - dI^{(k)}(0) \right), $$

$$ dI^{(k)}(\phi) = \lambda \sum_{m=1}^{M} \left( \frac{m \epsilon m^{(k)}(I_{0})}{2 \sin(\frac{m\mu}{2})} e^{im(\mu/2 + \phi^{(k)}) + \pi/2} + c.c. \right) $$

(14)

Notice that, to first order, the argument in $c_{m}^{(k)}$ has been replaced with $I_{0}$. We have also separated the two sums so that $dI^{(k)}(\phi) - dI^{(k)}(0)$ is the individual contribution of the $k$th IP. In the same way, a different initial condition may be used (more suitable for plots): $I(0) = I_{0}$, instead of $I(\pi/2) = I_{0}$.

The smear $S(n_{\sigma})$ is now defined as the normalized r.m.s. of the invariant, i.e. $\sqrt{V}$ with $V$ being the variance:

$$ S(n_{\sigma}) = \sqrt{V(I)}, \quad V = \frac{1}{2\pi} \int (I - \langle I \rangle)^{2} d\phi, \quad \langle I \rangle = \frac{1}{2\pi} \int I d\phi. $$

**VERIFICATION WITH TRACKING**

As an example application, this section studies the very simple collision set that still posses all the symmetries with l.r. set at 8 sigma as depicted on Figure 2. Both IR5 and IR1 are included. The goal here is to test the invariant $I(\phi)$ by tracking with a simple model built with kicks $\Delta x_{\text{non}}^{m}$ alternating with linear matrices and Sixtrack. The parameters are: energy $3.5\text{ TeV}$, $N_{0} = 1.210^{11}$ and normalized emittance $\epsilon_{n} = 2.510^{-0}$. Tracking single particles at various amplitudes with the simple model produces the results shown on Figure 3. A particle starts with $n_{\sigma} = 3$, or 7 ($I_{0} = 4.5$, or 24.5). The $c_{m}$ are computed with accuracy $10^{-7}$ – the value of $M$ in (15) as about 40. Since the beam potential changes the linear optics, we need to find the linearly-perturbed matched $\beta$-function value at the initial point for tracking. For the plots on Figure 3 this is done in a separate run using a linear kick $\langle \Delta x' \rangle_{\text{lin}}$ (only term $\sim x^{2}$ in the Hamiltonian). This is similar to what is done in Sixtrack. The resultant matched $\beta$ is used to define the initial coordinate $x_{0}$ (through $n_{\sigma}$). The values of the smear are shown on the top of each plot.

Figure 2: Sample setup: three collisions in each IR5 and IR1. The l.r. are set at 8 sigma.

Figure 3: Invariant tested on a simple kick-matrix model. Black points: turn-by-turn coordinates $(\phi, I)$ for $10^{3}$ turns. Red: invariant $I(\phi)$ (initials chosen so that $I(\pi/2) = I_{0}$).
Plotting the smear over a range of amplitudes with all three methods: model, Sixtrack and analytic $S(n_\sigma)$ results in Figure 4. Notice that here the images of the strong-beam centroids (see Fig. 1) are represented with vertical gray lines drawn at 0 and 8 sigma. Let us now look at the individual contributions to $I(\phi)$ of the six IPs at three amplitudes chosen arbitrarily, say $n_\sigma = 1, 3$ and 7. The excursions (w.r.t. $I_0$) of the individual invariant surfaces are shown on Fig. 5. Here $I(0) = I_0$. The color code is as on Fig. 2 and in addition for the head-ons we use solid black for IP5 and dashed for IP1. Near the axis ($n_\sigma = 1$), only the two head-ons contribute and, being of opposite signs, nearly compensate each other. At $n_\sigma = 3$, one begins to see long range contributions that grow when $n_\sigma = 7$. At such large amplitudes, the compensation is no longer true. Magenta and green are barely seen, meaning that contribution of off-plane collisions is negligible. Thus in case of test particle moving in the horizontal motion, the contribution of all l.r. in IR1 can be neglected, and vice versa for vertical motion and IP5.

Figure 4: Agreement with Sixtrack.

Figure 5: Individual contributions $dI^{(k)}(\phi) - dI^{(k)}(0)$ – color code as on Fig. 2

Above some critical strength of beam-beam interaction, i.e. quantities $N_{l.r.}$ and/or $N_l$ and/or inverse crossing angle, the first-order theory is no longer an adequate description of the smear. However, as we will see, the behavior of $S(n_\sigma)$ still may be used as an indication for dynamic aperture since it exhibits a local maximum near it. What happens is that the linear behavior, i.e. agreement between the first-order $S$ and Sixtrack at all amplitudes seen on Figure 4 is replaced by what is shown on Figures 6 a), b) and c). The blue ($S(n_\sigma)$) and the red (Sixtrack) curves depart from each other once $n_\sigma$ approaches amplitudes near the strong beam core, represented by the cluster of vertical gray lines. At this point the exact smear (red) exhibits a steep growth, thus the dynamic aperture is likely to be close to this point while $S$ goes through a maximum and then through a minimum, thus forming a dip. Upon exiting the core, past the last gray line, the red and blue curves nearly remerge. It can be shown that the above property of $S(n_\sigma)$ is a consequence of the left-right symmetry of IR5 and IR1. Namely, the individual contributions (such as red and blue curves on Fig. 2) change sign or flip about the axis each time $n_\sigma$ crosses a gray line. At this amplitude $S(n_\sigma)$ stops growing and goes through a maximum.

Figure 6: Various combinations of number of long-range collisions and and bunch intensity to illustrate linear and nonlinear behavior:

<table>
<thead>
<tr>
<th>$N_{l.r.}$</th>
<th>$N_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 10</td>
<td>$1.2 \times 10^{11}$</td>
</tr>
<tr>
<td>b) 32</td>
<td>$1.2 \times 10^{11}$</td>
</tr>
<tr>
<td>c) 32</td>
<td>$1.6 \times 10^{11}$</td>
</tr>
<tr>
<td>d) 32</td>
<td>$0.2 \times 10^{11}$</td>
</tr>
</tbody>
</table>

BEHAVIOR OF THE SMEAR $S(n_\sigma)$ NEAR THE DYNAMIC APERTURE
ANALYSIS OF LONG RANGE EXPERIMENTS

Dependence on intensity and crossing angle

We set the parameters as at the MD: energy 3.5 TeV,
$\epsilon_n = 2.5 \times 10^{-6}$, $\beta^* = 0.6$ m.

Of all collisions sets used at the MD, let us consider three
$N_{l.r.} = 32, 24$ and 16. For each of them, two parameters:
bunch intensity $N_b$ and (half) crossing angle $\alpha$ uniquely
define the dependence the first-order smear on amplitude
$S(n_\sigma; N_b, \alpha)$ through the following procedure. First, be-
ing a first-order quantity in $\lambda$, the smear is obviously pro-
portional to the intensity: $S \sim N_b$. Secondly, the depen-
dence of $n^{(k)}_{x,y}$ on the (half) crossing angle $\alpha$ is given by
the well known scaling law:
$n^{(k)}_{x,y} \sim \alpha \sqrt{\beta^*}$, where $n^{(k)}_{x,y}$ are taken from some sample lattice built for
$\beta^* = 0.55$ m and $\alpha = 125$. Finally, the phases $\phi^{(k)}$ are assumed to be
independent on $\alpha$.

The dependence on angle is presented on Figure 7. Each
blue branch corresponds to $S(n_\sigma; 1.2 \times 10^{11}, \alpha)$ taken over
an amplitude range where it is monotonically increasing
hence, as we already know, it will remain in agreement with
tracking for any strength of the beam-beam interaction.

Coming now to the MD, the observed losses during re-
duction of the crossing angle in IP1 are shown on Figures 9
and 10 [1].

\[ S(n_\sigma; 1.2 \times 10^{11}, \alpha_1) = S(n_\sigma; 1.6 \times 10^{11}, \alpha_2), \quad (15) \]

which is to be solved for the angles.

\[ \alpha_1 \approx 87 \, \mu \text{rad} \]

\[ \alpha_2 \approx 96 \, \mu \text{rad} \]

Figure 7: Dependance of the smear(amplitude) graph on
the parameter $\alpha$ for $N_{l.r.} = 32$ (top) and $N_{l.r.} = 24, 16$
(bottom). Each graph is restricted within a domain ex-
tending up to its first maximum (red dot) (entrance into the
strong-beam core).

Figure 8: The collision sets for $N_{l.r.} = 32$ and 24 are built
by dropping the first and last 2 or 4 elements from the full
set ($N_{l.r.} = 32$).

Figure 9: Experiment with $N_b = 1.2 \times 10^{11}$: losses start at
$\alpha_1 \approx 87 \, \mu \text{rad}$

Figure 10: Experiment with $N_b = 1.6 \times 10^{11}$: losses start at
$\alpha_2 \approx 96 \, \mu \text{rad}$

Explanation of the case $N_{l.r.} = 32$ (brown curves)

For $N_{l.r.} = 32$ (the full 50-ns collision set shown on Fig-
ure 8) we need to explain the brown curves on Fig 9 and 10. Here
losses are seen to start at $\alpha_1 \approx 87$ and $\alpha_2 \approx 96 \, \mu \text{rad}$
respectively.

In view of our previous findings, the off-plane losses (in
IRS) are neglected and by using the postulate made in the
Introduction, Eqn 1, we have:

\[ S(n_\sigma; 1.2 \times 10^{11}, \alpha_1) = S(n_\sigma; 1.6 \times 10^{11}, \alpha_2), \quad (15) \]

which is to be solved for the angles.

\[ \alpha_1 \approx 87 \, \mu \text{rad} \]

\[ \alpha_2 \approx 96 \, \mu \text{rad} \]
The resolution of Fig 9 and 10. By looking now at the bottom two plots in Fig 9 and 10. Smear, 1.5

Fig 11 demonstrates that a good solution to (15) are angle values \( \alpha_1 = 86 \), \( \alpha_2 = 96 \) \( \mu \)rad. Indeed, this Figure shows that (15) is fulfilled not in a single point, but for all amplitudes up to 1.5 \( \sigma \) where the smear reaches about \( \approx \) 3%. What has happened of course is that scaling by a factor 1.6/1.2, but reducing the angle from \( \alpha_2 \) to \( \alpha_1 \) has nearly exactly preserved one particular blue branch from Figure 7. Conversely, small variations about this solution, say \( \pm 5 \) \( \mu \)rad, lead to red and blue curves deviating from each other as this is seen on Figure 12.

Figure 11: Graphs of \( S(n_{\sigma}; 1.2 \times 10^{11}, 86) \) (red) and \( S(n_{\sigma}; 1.6 \times 10^{11}, 96) \) (blue). The smear is seen to be \( \approx 3\% \) at 1.5 \( \sigma \).

Figure 12: Small variations about the solution \( \pm 5 \) \( \mu \)rad

Explanation of cases \( N_{l,r} = 24 \text{ and } 16 \) (green and black)

For \( N_{l,r} = 24 \text{ and } 16 \) (reduced collision sets on Figure 8) one needs to explain the green and black decay curves on Fig 9 and 10. By looking now at the bottom two plots in Figure 7, we seek for blue branches that pass through the same maximum-smear point as found above: 3% at 1.5 \( \sigma \). The resultant branches are plotted on Figures 13 and 14 with solution angles summarized in Table 1. Again, at least a qualitative agreement is observed to the extent allowed by the resolution of Fig 9 and 10.

Note that, for the four plots in Figures 13 and 14, on three occasions the 3%-smear line intersects a monotonic part of \( S(n_{\sigma}) \) where, as we already know from Sect. 4, there is an agreement with Sixtrack. The location of the maximum of \( S \) (dashed vertical line) is thereby not used to determine the angle. For the remaining case, \( \alpha = 53 \), the location of the maximum has been used as an indicator of dynamic aperture.

Table 1: Solutions for the crossing angle in case of green and black decay curves – Figures 9 and 10.

<table>
<thead>
<tr>
<th>( N_0 )</th>
<th>green</th>
<th>black</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.2 \times 10^{11} )</td>
<td>65</td>
<td>53</td>
</tr>
<tr>
<td>( 1.6 \times 10^{11} )</td>
<td>83</td>
<td>72</td>
</tr>
</tbody>
</table>

REFERENCES

[1] R. Assmann et al Results of long-range beam-beam studies - scaling with beam separation and intensity, CERN-ATS-Note-2012-070 MD
The nonlinear kick-factor (10) is
\[ F_{\text{nonl}}(\sigma, \phi) = \gamma + \Gamma(\sigma, \phi) \equiv \left( \frac{2 n_2}{(n_2^2 + n_2^4)} \right)^{1/2} \left( 1 - e^{\frac{-2 n_2}{n_2^2 + n_2^4}} \right) \sin \phi \]
\[ F^{(1)}(x) = \frac{2 n_2}{(n_2^2 + n_2^4)} \left( 1 - e^{\frac{-2 n_2}{n_2^2 + n_2^4}} \right) \sin \phi, \]
\[ F^{(2)}(x) = \left( n_2^2 + n_2^4 \right)^{1/2} \left( 1 - e^{\frac{-2 n_2}{n_2^2 + n_2^4}} \right) \sin \phi. \]

By following [4], the Lie map is given by an expression of the form:
\[ M_{N+1}e^{f^{(N)}(\sigma)} \cdots e^{f^{(2)}(\sigma)} M_2 e^{f^{(1)}(\sigma)} M_1 = f^{(k)}(x) \equiv F_{\text{nonl}}(x). \]

Here, the maps are linear operators and for brevity, we have replaced \( F_{\text{nonl}}(x) \) with \( f^{(k)}(x) \). We will show that since \( F_{\text{nonl}} \) depends only on the normalized coordinate \( x/\sigma \), once we rewrite it in terms of the eigen-coordinates at the \( k \)th kick, the local beta-functions \( \beta(k) \) disappear while the phase \( \phi(k) \) is simply added to \( \phi \).

By reversing the order, the map transforming the test particle \((x, p_x)\) for one turn around the ring is
\[ \mathcal{M} = M_1 e^{f^{(3)}(\sigma)} M_2 e^{f^{(2)}(\sigma)} \cdots M_N e^{f^{(N)}(\sigma)} M_{N+1} = e^{- \mathcal{A}_1 f^{(1)}(\sigma)} e^{- \mathcal{A}_2 f^{(2)}(\sigma)} \cdots e^{- \mathcal{A}_N f^{(N)}(\sigma)} \mathcal{M}_{N+1}. \]

Reversal of the order means that in the first line all \( f^{(k)} \) are now functions of the same initial variables \((x, p_x)\). In the second line, accumulated linear maps \( \mathcal{M}_k = M_1 M_2 \cdots M_k \) have been applied to transform the initial vector to the kick location. Thus, as a first step, we have moved all kicks to the front of the lattice and \( \mathcal{M}_{N+1} \) is the total one-turn linear Lie operator.

Let us denote matrices corresponding to Lie operators with hats, e.g. \( \mathcal{M}_{N+1} \). As a second step, with \( \beta, \alpha \) being matched Twist parameters at the end of the lattice, one uses an \( A_0 \) transform that transforms the ring matrix to a rotation (inserting identities \( A_0 A_0^{-1} \) in between the exponents):
\[ \mathcal{M}_{N+1} \xrightarrow{\mathcal{A}_0} \hat{R} = \begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix}, \]
\[ \mathcal{A}_0 = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha \sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \]

The two steps above combined are equivalent to replacing the argument of \( f \) by \( \hat{x}_k - \) the eigen-coordinate at the \( k \)th location. To see this, apply the \( \mathcal{A}_0 \) transform to both kick-factor and coordinate:
\[ A_0 \mathcal{M}_k f^{(k)}(x) = f^{(k)}(A_0 \mathcal{M}_k x) = f^{(k)}(\hat{x}_k), \]
\[ \hat{x}_k \equiv A_0 \mathcal{M}_k x = \sqrt{2 \beta(k)} \sin (\phi + \phi(k)). \]

One can now drop the \( \mathcal{A}_0 \) on both sides of \( M \) and consider the map:
\[ \mathcal{M} = e^{\hat{f}^{(1)}}(J, \phi) \cdots e^{\hat{f}^{(N)}}(J, \phi) \equiv R \]
\[ \hat{f}^{(k)}(J, \phi) = f^{(k)}(\hat{x}_k) \]
\[ R = e^{f_2}, \quad f_2 := -\mu J. \]

To first order, one can just sum the Lie-factors:
\[ \mathcal{M} \approx e^{F}, \quad R = e^{h}, \quad F \equiv \sum_{k=1}^{N} \hat{f}^{(k)}. \]

By noticing that above, as in [4], \( R \) precedes the kick, while in (2) and [3] the kick is assumed to be at the end of the lattice, our map is identical to (2).

The first order invariant \( h \) is now found with the BCH theorem. Let us write \( F = F + F^* \) where \( F^* \) is the oscillating part. By taking only \( F^* \):
\[ h(J, \phi) = f_2 + \frac{\mu}{1 - e^{-f_2}} F^* \quad \text{(A.2)} \]
\[ F^* \equiv \sum_{k=1}^{N} \hat{f}^{(k)}(J, \phi) \]
where according to (12)
\[ \hat{f}^{(k)} = \sum_{m \neq 0} C_m^{(k)} e^{i m \phi} = \sum_{m=1}^{\infty} C_m^{(k)} e^{i m \phi} + cc \]

A basic property of \( f_2 = \) is to operate in a simple way on functions of \( J \), or eigen-vectors \( e^{im\phi} \). Also functions \( G(f_2) \) can easily be applied to eigen-vectors:
\[ G : f_2 = e^{i m \phi} = (i n \mu e^{i m \phi}). \]
\[ G : f_2 = e^{i m \phi} = (i n \mu e^{i m \phi}). \]

If we choose \( G : f_2 = \equiv \frac{f_2}{1 - e^{-f_2}} \), then we have:
\[ G(f_2) = e^{i m \phi} = e^{i n \mu e^{i m \phi}} - e^{-i n \mu e^{i m \phi}} \]
\[ = \frac{i n \mu e^{i m \phi} - e^{-i n \mu e^{i m \phi}}}{e^{i n \mu e^{i m \phi}} + e^{-i n \mu e^{i m \phi}}} \]
\[ = \frac{e^{2 i n \mu e^{i m \phi}}}{2 \sin (n \mu/2) e^{i n \mu e^{i m \phi}}}. \]
By substituting all these in (A.2) and using the property
\( C_m^{(k)} = c_m^{(k)} e^{i m \phi^{(k)}} \) we get:

\[
h(J, \phi) = -\mu J - \lambda \sum_{k=1}^{N} \sum_{m=1}^{\infty} \left( \frac{m \mu c_{m}^{(k)}}{2 \sin (m \mu/2)} e^{im(\mu/2+\phi+\phi^{(k)})} + c.c. \right)
\]